Example problem

Level: Middle secondary

Intermediate modelling

(IM²C international mathematical modeling challenge)

Bushwalking

Describe the real-world problem

It is not uncommon for companions who enjoy bushwalking to differ in fitness and energy. On tracks that lead out and back they will often walk together for a time at the pace of the slower walker, until the slower walker indicates an intention to turn around and return to base.

The faster walker has the choice of following the same action, but alternatively may decide to carry on for a time at a faster pace before also returning. Especially if the opportunity to travel further and faster is appreciated, the faster walker will want to go as far as possible.

However, they will not want their companion to have to wait around too long at the end of the walk for them to return.

Specify the mathematical problem

When the slower walker starts the return trip, for how much extra time should the faster walker travel on the outward path before turning for home, so that both will arrive at the starting point at the same time?

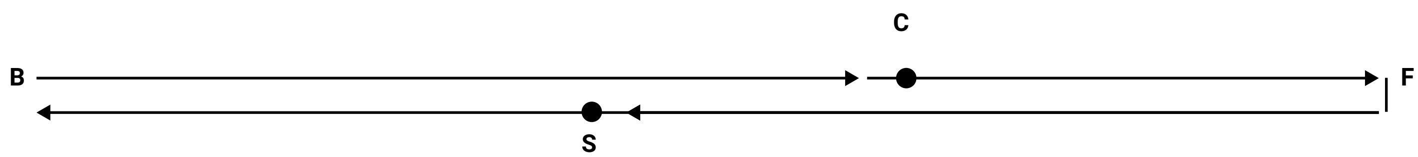
Formulate the mathematical model

See the figure modelling the problem.

Let B = starting point and C = point that walkers reach together before separating.

Suppose that S is the point reached by slower walker (s) on the homeward path when faster walker (f) turns for home at F.

BF and FB are outward and return paths of the faster walker along the same track.



Assumptions

While together, both proceed at the speed of the slower walker. After parting, both walkers maintain their respective average walking speeds.

Let t = time taken to reach C while walking together at the speed V of the slower walker.

Slower walker (s) now starts to return while the faster walker (f) continues on as far as F at speed (kV) for an additional time T where k > 1.

For the walkers to arrive at B together:

time taken for f to cover the distance FB = time taken for s to cover the distance SB.

So FB/kV = SB/V

Solve the mathematics

BC = Vt; CF = (kV)T; CS = VT

FB = BC + CF = Vt + kVT

SB = BC – CS = Vt – VT

Hence V(t + kT)/kV = V(t – T)/V

So (t + kT)/k = (t – T)

2kT = (k – 1)t

T = [(k – 1)/2k]t

T can now be found by substituting different values for t, V, and k.

(It is common for the formulation of a model to merge seamlessly into the mathematics of its solution.)

Interpret the solution

It is sufficient to know the time from the start of the walk to the separation point (t), and the relative walking speeds (k) of the two individuals. For the first we need to remember to look at a watch at the start, and when point C is reached. (Note that spending some time at C before resuming the walk will not affect the outcome.) And relative speeds are easily estimated by comparing times taken to cover a chosen distance.

Firstly, check the mathematics for sensible outcomes.

k = 1 gives T = 0

Both walkers turn together if they walk at the same pace, as should happen.

Suppose walkers stay together for an hour.

t = 1 so T = (k – 1)/2k

Suppose k = 2: T = $\frac{1}{4}$ ('f' should continue on for 15 minutes).

This seems a sensible figure.

Secondly, in terms of the real context, the outcome is amenable to direct checking. Try it out having estimated a value for k. This helps to underline that real-world problem solving cannot live entirely in a classroom.

Evaluate the model

During evaluation of the solution to an original problem, a related problem at times suggests itself, and stimulates another application of the modelling cycle or parts of it. In this case we might consider that it is more realistic in practical terms for 'f' to aim to arrive back at base so that the slower walker doesn't have to wait 'too long' for the faster walker to return. That is, a modified modelling question is set in terms of a time window 'w', so that
0 < (difference in arrival times) < w.

Then in terms of formulating the model and solving the mathematics we have the following.
After the walkers separate, 'f' travels a distance (2CF + CB) at speed (kV) to reach base B.
Time for 'f' to reach base = (2kVT + Vt)/kV = (2kT + t)/k.

After the walkers separate, 's' returns to base B at speed V (same as outward trip).
Time for 's' to reach base = t.

So 'f' reaches base at a time
[(2kT + t)/k -t] = [2kT – (k – 1) [(k – 1)t + kw)]/2k t]/k
after 's' has arrived.

So we need 0 < [2kT – (k-1) t]/k < w

That is [2kT – (k-1) t]/k > 0 and [2kT – (k-1) t]/k < w

[2kT – (k – 1) t]/k > 0

gives T > t(k – 1)/2k

and [2kT – (k – 1)t]/k < w

gives T < [(k – 1)t + kw)]/2k

Hence t(k – 1)/2k < T < [(k – 1)t + kw)]/2k

For example, if w = 0.2 (12 minutes) and as before we consider the case where

k = 2, t = 1, then we obtain 0.25 < T < 0.35.

Interpreting, we have that if 'f' aims to have 's' wait no longer than 12 minutes at the starting point at the end of the walk, f should continue for about 15 to 20 minutes from the point where the walkers separate.

Report the solution

The modelling report would contain all the above components of the modelling problem and its solution synthesised into a cohesive narrative. Additional working or explanations can be added when judged to enhance the product, with its completeness assessed in terms of, for example, the checklist on report writing.