Example problem

Level: Middle and senior secondary

Senior modelling

(IM^{2}C) international mathematical modeling challenge

Population growth

Describe the real-world problem

Australia's population to hit 23 million

Tuesday, 23 April, 2013.

Australia's population will reach 23 million people overnight, and is on track to surpass 40 million within 40 years.

Projections build on last year's known population, and take into account fertility, life expectancy and immigration figures. Figures from the Australian Bureau of Statistics show that around 180 000 people move to Australia each year.

The ABS estimates that there is a birth every one minute and 44 seconds, a death every three minutes and 32 seconds, and a new migrant arriving every two minutes and 19 seconds.

That means our population increases by one person every minute and 23 seconds – more than 1000 people per day.

Specify the mathematical problem

Problem a: Daily population growth

Simple problem. Check and comment on the claim that 'our population increases by one person every minute and 23 seconds – more than 1000 people per day.'

This amounts to evaluating claims made by others on the basis of someone else's model. This is itself an important activity to recognise and undertake when appropriate.

Problem b: Population growth over 40 years

Advanced problem: Is it likely that Australia's population will reach 40 million in 40 years?

Formulate the mathematical model (a) for daily growth

Increase in persons per day = no of births per day – no of deaths per day + no of net migrants per day.

Solve the mathematics (a) for daily growth

As on 23 April 2013:

no of births per day = $(24 \times 60 \times 60)/104 = 830.77$
(one birth every 104 sec)

no of deaths per day = $(24 \times 60 \times 60)/212 = 407.55$
(one death every 212 sec)

no of (net) migrants per day = $(24 \times 60 \times 60)/139 = 621.58$
(one migrant arrives every 139 sec)

So, increase in persons per day = 830.77 − 407.55 + 621.58
= 1044.8 (1045 persons)

Time interval between arrivals = (24 × 60 × 60)/1045
= 82.68 sec (1 min 23 sec approx)

Interpret the solution (a) for daily growth

The calculations verify the claims that on this day the population increases by one every 1 min 23 sec, which is more than 1000 persons per day

Evaluate the model (a) for daily growth

The model is built in terms of births, deaths, and net immigration, and gives results precisely in line with the claims. We can be confident it is a good model for the purpose it was created.

However, we can point to reservations about its wider use. The model uses data for births, deaths, and net immigration that are applicable on a particular day. For population predictions long-term, we need annual estimates of these quantities. For example, a migration rate of 621 per day (the figure that applies on 23 April 2013) if used to calculate an annual figure gives a value over 226 000 (many more than the average of 180 000, given by the ABS).

The challenge of estimating population size into the future is considered in the next problem.

Formulate the mathematical model (b) for growth over 40 years

The advanced problem asks: Is it likely that Australia's population will reach 40 million in 40 years?

Given that births, deaths, and net migration have been established as the key variables, the emphasis changes to designing a model for long-term population forecasts – typically expressed in terms of *years*.

It is useful to summarise what we know from the data given in the report (see the table). The last column generalises the calculations to express the *yearly* change in a population (P) in terms of the respective contributions from births, deaths, and migration.

The calculations assume that the data given for April 23 apply across the year, which seems to be a suggestion within the reporting. This is a point for discussion in its own right.

Assumptions

For our first model we will use the values given in the report, and included in the table. This assumes that the fractional birth rate, the fractional death rate, and the rate of migration remain the same over the time scale of the model.

Note that in published statistics, migration rate refers to the net rate of increase through migration.

These assumptions deserve discussion, as they provide a means of making progress, but may also be a source of reservation in accepting predictions. They will be revisited when evaluating model outcomes.

Population in 2013 = 23 000 000					**Population = P**
Births	one per 104 seconds	births per day = (60 × 60 × 24) ÷ 104 = 830.77	births per year = 830.77 × 365.25 = 303 439	Fractional birth rate (b) = births per year ÷ total pop = 303 439 ÷ 23 000 000 = 0.0132 per year (yr^{-1})	births per year = Pb
Deaths	one per 212 seconds	deaths per day = (60 × 60 × 24) ÷ 212 = 407.55	deaths per year = 407.55 × 365.25 = 148 856	Fractional death rate (d) = deaths per year ÷ total pop = 148 856 ÷ 23 000 000 = 0.00647 per year (yr^{-1})	Deaths per year = Pd
Net migration	one per 139 seconds	migration per day = (60 × 60 × 24) ÷ 139 = 621.58	migration per year = 621.58 × 365.25 = 227 032	Migration numbers (m) = 227 032 persons/year	Migrants per year = M/365.25 where M = total migrants in a year
Note that net migration is independent of current population, while the number of annual births (Pb) and annual deaths (Pd) is directly influenced by it.					

Real-world material relevant to both modelling and discussion can be obtained from internet sources. Students will often do this on their own initiative, or can be prompted if relevant. (It is important that such activity remains focused on the modelling task – for example, is related to clarifying assumptions – and does not become a diversion).

Solve the mathematics (b) for growth over 40 years

For this first model, we use the data provided in the stimulus text, and summarised in the table. There are a number of ways this problem can be approached mathematically, and three different approaches are shown.

The spreadsheet solution can be applied by anyone with spreadsheet competence. The other solutions require knowledge of geometric series and calculus respectively, which locates them within senior mathematics.

Let P_0 = initial population (in 2013)

Let P_n = population in year n

Let r = natural population change rate
= birth rate (b) – death rate (d) i.e. (b – d)

Let M = average net annual immigration intake

To solve by spreadsheet

$P_0 = 23\,000\,000$; $b = 0.0132$; $d = 0.00647$; $r = b - d = 0.00673$;
$M = 227\,032$

$P_1 = P_0 + rP_0 + M = P_0 (1 + r) + M = P_0 R + M$ (where $R = 1 + r$)

$P_2 = P_1 R + M$ etc

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$P_{40} = 40\,459\,252$

To solve by geometric series

Proceeding as above (annual increments)

$P_1 = P_0R + M$

$P_2 = P_1R + M = P_0R^2 + M (R + 1)$

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$P_n = P_0R^n + M (R^{n-1} + \ldots R^2 + R + 1)$

$P_n = P_0R^n + M (R^{n-1})/R - 1$

$P_{40} = 40\,459\,252$ (annual increments)

$P_{40} = 40\,523\,429$ (monthly increments)

$P_{40} = 40\,523\,960$ (daily increments)

To solve by calculus

$\delta P \approx Pr\,\delta t + M\,\delta t$

$= (Pr + M)\,\delta t$ where $r = b - d$.

In the limit

$dP/dt = Pr + M$

$\ln(Pr + M) = rt + \text{constant}$, leading to

$P(t) = P_0e^{rt} + M(e^{rt} - 1)/r$ where $P(0) = P_0$

$P_{40} = 40\,526\,191$

Interpret the solution (b) for growth over 40 years

All methods confirm a predicted population of around 40 million in 40 years' time.

Evaluate the model (b) for growth over 40 years

Appropriate mathematical growth processes have been applied, and similar results obtained using three different methods. It seems reasonable to trust the methods.

What about the assumptions leading to the values used for b, d, and M which were all based on their values on a particular day: April 23, 2013?

It is easy for death rates to have a 'spike' – for example during 'flu epidemics, or for some reason to be lower than average during a short interval. For predictions, we need stable average values. If members of a species have an average life time of 10 years, then on average 1/10 of a population (the fractional death rate) will die each year. Generalising, an average lifetime of 'L' means that on average a fraction '1/L' of the population die each year. In the fifth column of the table we see that $d = 0.00647\ yr^{-1}$, which implies an average life time of 154.6 years.

Clearly the value on April 23 2013 was not typical and we need a more representative value.

Websites such as the Australian Institute of Health and Welfare give figures for life expectancy for Australians (http://www.aihw.gov.au/deaths/life-expectancy).

Life expectancy is not quite the same as average lifetime but is a close approximation (an interesting point of discussion if time is available). If the average life time in 2013 was 81.5 years, the average fractional death rate = 1/81.5 = $0.0122699\ yr^{-1}$.

Also, the net immigration rate is quoted officially at about 180 000 per year, which is substantially less than the figure implied by the specific value that applied on the special date.

Leaving the value of 'b' alone, which is essentially determined by personal family planning decisions, and adjusting the values of 'd' and 'M' as above, show that using the above three methods of calculation, leads to a value of P_{40} = 31 200 000 (approximately).

This is substantially less than the previous estimate. There are implications for jobs, housing, health provision, education… if forecasts are relied upon. This adds a social context around the problem.

Refinement (advanced)

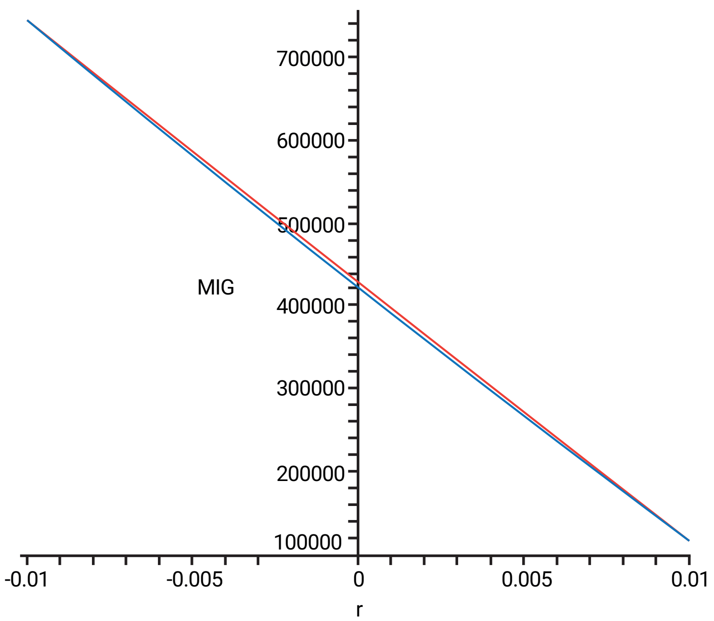
Over time, the natural growth rate (births – deaths) is monitored and published as statistics. Knowing these, immigration can be used as a policy tool to address population issues. We can use this idea to pose a different question.

Suppose a population of 40 million is the goal for 2053 (40 years from 2013). What combination of natural growth rates and immigration rates would achieve this?

For this senior level refinement using $P = P_0 e^{rt} + M(e^{rt} - 1)/r$ from previous work we need

$40\,000\,000 = 23\,000\,000\, e^{40r} + (e^{40r} - 1)I/r$

This problem requires facility with CAS technology such as Maple or Mathematica. Using the former and noting that MIG = M the graph is shown on the axis below (blue in online format).



An almost identical graph of a linear approximation has equation M = 430 652 – 31 500 000r (red). Now feasible solutions can be considered by noting the real world meanings of r and M. For example if r = 0 (so called replacement rate when deaths are just matched by births) then M = 430 652, leading to a population of 40 226 080 in 2053. Is this realistic? Allowing 'r' and 'M' to vary within the 40 year period provides for considering implications of changes in family planning decisions (personal) and immigration policy (national).

Report

The modelling report should contain all the components of the modelling problems, their solutions, interpretation and evaluation. The report could provide a cohesive narrative, including some discussion of implications of Australian population growth for policy, society and economy around matters such as jobs, housing, health, and education. When forecasts are astray where does responsibility lie? With demographers? With data problems? Has the media been creative with facts or interpretations?