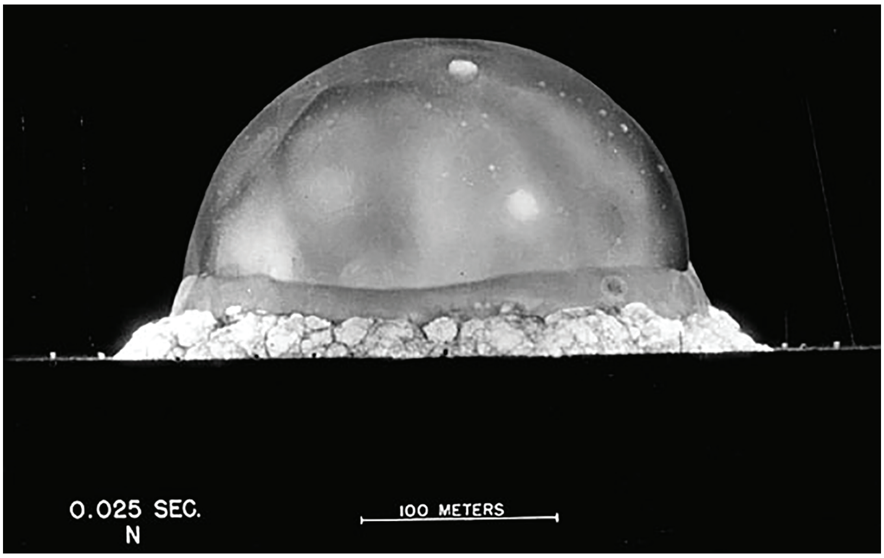
Example problem

Level: Senior secondary

Self-generated modelling

international mathematical modeling challenge

Nuclear blast



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Describe the real-world problem

The likely properties of the planned first atomic bomb were unknown until a trial detonation, the Trinity atomic test, was undertaken in the New Mexico desert in 1945.

Among the invited witnesses to the trial was a Cambridge Professor, Geoffrey Taylor, who had been an advisor to the Manhattan project team, the group responsible for developing the nuclear device. Simply witnessing the trial did not provide a measure of its strength.

Later, in 1947, photographs of the blast were made public in a variety of sources, including *Life* magazine. Taylor was browsing through one of these which contained a report of the test, together with photos of the expanding blast wave, taken over a succession of small time intervals. Armed with a series of photos, he devised a question and conducted modelling to estimate the energy released in the blast.

Taylor's approach to the problem has been discussed in various sources, including by University of Cambridge Professor Timothy Pedley in the journal *Mathematics Today* in 2005. The paper describes (with illustrations) how to apply mathematics to real-world problems, including a simplified working of the nuclear blast problem.

Sufficient information is provided to enable the investigation to be illustrated through a modelling approach.

Specify the mathematical problem

Estimate the energy released in the bomb blast.

Formulate the mathematical model

This problem illustrates two significant aspects about modelling activity. Firstly it illustrates an important attribute of a modeller – the identification by modellers of model-rich situations in the first place. Curricular goals for students to use their mathematics to solve problems of personal interest, in work contexts, and as productive citizens require this ability. (The International Mathematics Modeling Challenge program does not explicitly address this aspect, as by its nature problems are specified as the starting point.) Secondly, this problem affirms the use of a cyclic modelling process by a professional modeller.

The fundamental assumption is that the radius of the spherical blast wave (R) depends on a product of three factors: the time elapsed since the explosion (t), the instantaneous energy released (E), and the density of air (ρ). Thus $R = Ct^aE^b\rho^c$, where C is a dimensionless constant.

(These assumptions are taken as a given for this development. Their plausibility can be discussed if desired.)

Solve the mathematics

Taylor then proceeded to apply dimensional analysis as elaborated below, in order to find suitable values for the indices a, b and c, and to use the results to estimate the amount of energy released.

Using the standard notation used for dimensions dim R = [R] etc, to express quantities in terms of the fundamental dimensions mass (M), length (L), and time (T) we have:

$[R] = L$, $[t] = T$, $[E] = ML^2 T^{-2}$, and $[\rho] = ML^{-3}$

Thus dimensionally, using the formula above, we need:
$L^1 = M^{(b + c)} L^{(2b - 3c)} T^{(a - 2b)}$

Equating dimensions on both sides of the equal sign we need : $b + c = 0$; $2b - 3c = 1$; $a - 2b = 0$

hence $a = \frac{2}{5}$, $b = \frac{1}{5}$, and $c = -\frac{1}{5}$.

So $R = C\,(Et^2/\rho)^{\frac{1}{5}}$

(A value of $C \approx 1$ was assigned on the basis of knowledge of blast activity, and hence $E = \rho R^5/t^2$.)

The above photograph of the blast contains a scale representing 100 m, and a label indicating that it was taken at t = 0.025 (s). Expanding the photo from the link, taking measurements, and using the given scale to estimate the radius suggests a value for R of about 132 m.

Noting that density of air is 1.2 kg/m^3 and substituting in the above formula gives a value for E of about 7.7×10^{13} joule.

Interpret the solution

The standard measure of strength of bomb blasts is given as a comparison with the yield of the explosive material TNT, trinitrotoluene. Internet research indicates that the energy released by 1 metric tonne of TNT is equivalent to about 4.148 gigajoules, where 1 gigajoule equals 1 billion joules. Check that the value of E for the Trinity blast calculation (using the single diagram above) converts to an energy equivalent of about 18.4 kilotonnes of TNT.

Evaluate the model

The method used by Taylor was slightly different from the above which is based on measurement and calculation from one photograph. Taylor used data from the series of photographs to plot the graph of log R against log t.

(Note that $R = (Et^2/\rho)^{\frac{1}{5}}$ can be written as
$\log R = \log (E/\rho^{\frac{1}{5}}) + 2 \log t$, so that the graph of log R against log t is linear. The value of E can then be robustly estimated from the intercept on the vertical axis). This is illustrated briefly in the Pedley article.

An internet search will identify many sources that discuss aspects of the Taylor approach. Several of these provide a series of photographs of the blast wave at successive time intervals. These can be used to give separate estimates of the energy released (as in the calculation above), or together to generate a log-log plot as used by Taylor. The sources do need vetting before being given the students, as some include unnecessary complications.

A useful source (Codoban, 2004) is located at http://www.atmosp.physics.utoronto.ca/people/codoban/PHY138/Mechanics/dimensional.pdf

Report the solution

Despite post-war national security issues, Taylor published his results showing the Trinity atomic bomb had a power equivalent of about 17 kilotons of TNT. The US Army was annoyed – the information was supposed to be classified!

References

Codoban, S. (Ed.) (2004). *Estimate of the energy released in the first atomic bomb explosion* [tutorial notes for University of Toronto subject Physics for the Life Sciences]. http://www.atmosp.physics.utoronto.ca/people/codoban/PHY138/Mechanics/dimensional.pdf

Pedley, T.J. (2005). Applying Mathematics. *Mathematics Today*, 41(3), 79–83.

Taylor, G.I. (1950). The formation of a blast wave by a very intense explosion. II. The atomic explosion of 1945. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 201*(1065), 175–86. http://www3.nd.edu/~powers/ame.90931/taylor.blast.wave.II.pdf