Example problem

Level: Senior secondary

Harder modelling

Farm dams

Describe the real-world problem

Farmers get their feet wet

In a dry country like Australia, farm dams are part of the lifeblood of rural life. Knowing the volume of water at any time is important for planning the numbers and distribution of livestock, and estimating when the supply is likely to run out under drought conditions.

Methods of estimating the volume of a partly empty dam depend on interpreting physically observable signs. If depth markers are embedded when the dam is excavated, the volume can be estimated from the water level measured on the marker. If dams do not have markers, some other method is needed.

Dams lose water by evaporation from the water surface to the atmosphere. Annual average evaporation rates are estimated using data collected from locations throughout the country. They vary from about 100 cm in western Tasmania up to 400 cm in the desert regions of northern Australia. Values in Victoria vary from about 140 cm in the south to 180 cm in the north, according to the Bureau of Meteorology (http://www.bom.gov.au/watl/evaporation). Loss of water by seepage is negligible.

Livestock water requirements

The table, containing data from a primary industries website, shows water requirements for a variety of farm animals. It contains information that farmers would know for their own stock, and is provided as a resource for this problem.

Table 1 Livestock water requirements

Stock	Litres/animal/year
Sheep	
nursing ewes on dry feed	3300
fat lambs on dry pasture	800
mature sheep — dry pasture	2500
fat lambs — irrigated pasture	400
mature sheep — irrigated pasture	1300
Cattle	
dairy cows, dry	16 000
dairy cows, milking	25 000
beef cattle	16 000
calves	8000

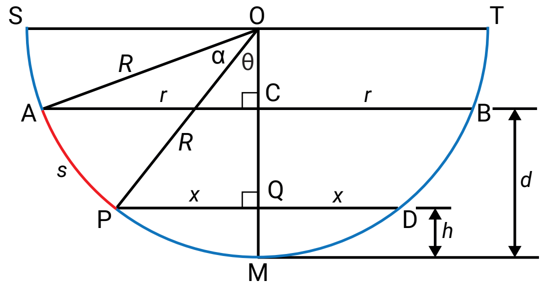

Figure 1 Cross-sectional diagram of a partly full dam

Specify the mathematical problems

The dam (Figure 1) has been excavated in the shape of an inverted spherical cap AMB – so that its cross-section is circular whatever the depth of water, and stock can drink from any point on its perimeter. The diameter of the dam when full (AB) is 30 metres, and its depth when full is 2 metres. The figure shows cross-sections of the dam, and of the hemisphere SMT of which it is a part. PMD is a representative section of the partly filled dam of radius ‘x’ and depth ‘h’.

Problem a: Model for dam volume

Find a way of estimating the volume of water in the dam when the (unknown) depth of water is ‘h’.

Problem b: Model for water loss over time

Suppose the dam is in central Victoria. If the dam is full and no more rain falls, how many days it would take for the dam to dry up?

Now suppose that a flock of 300 mature sheep on dry pasture is sustained by the dam. If the dam is full and no more rain falls, how long will it be before other arrangements need to be made for the stock?

Formulate the mathematical model (a) for dam volume

Discussion with students can be used to identify variables and measurements that will be needed to address the problem. Only data that can be accessed are useful, and the depth of water is not one of these. One readily obtainable piece of data is the distance the water surface has receded from its position when the dam is full — a distance obtained simply by stepping out the distance taken to walk from the edge of the dam, directly to the current water level. A key insight is the recognition that this distance (s) is both measurable, and a key input to the model.

Assumptions

- Water in the dam is reduced by evaporation and in providing for livestock. Seepage loss is negligible and can be ignored.
- Dams whose surfaces are bounded by curved perimeters and with access for animals can be represented by circular or elliptical approximations.
- The dimensions of the dam when empty (as excavated) are known.

Model for dam volume

The aim is to create a formula that enables the volume to be estimated in terms of the arc length AP(s) which we can measure by stepping to the edge of the water, from the high water mark when the dam is full. (The distance ‘s’ needs to be related to the depth of water ‘h’ which cannot be measured directly.)

The volume of the cap of a sphere of depth ‘h’ is given by the formula $V = \pi h^2(3R - h)/3$, where R is the radius of the parent sphere. (This result can be provided after students have identified a need for it, or can otherwise be obtained as an application of integral calculus.)

From Figure 1 we have from triangle OAC:

$R^2 = r^2 + (R - d)^2$ so $R = (r^2 + d^2)/2d$

$r = 15$ and $d = 2$ gives $R = 57.25$.

Then V (full dam) $= \pi d^2(3R - d)/3 = 710\text{m}^3$ (approx.) = 710 000 litres.

Similarly, from triangle OPQ:

$R^2 = x^2 + (R - h)^2$ so that $x^2 = 2Rh - h^2$

This links the radius of the water surface and depth, when the dam is partly filled, that is, for various values of ‘h’.

So we know how to calculate V when ‘h’ is known from $V = \pi h^2 (3R - h)/3$, but we need the volume in terms of a section of arc length ‘s’ that we can actually measure.

Hence we need to express ‘h’ in terms of ‘s’.

From Figure 1, we get $h = OM - OQ = R - R\cos\theta$.
Hence $h = R(1 - \cos\theta)$ … (1)
Now $\theta = (\theta + \alpha) - \alpha$ where $\theta + \alpha = \sin^{-1}(r/R) \approx 0.265$ and
$\alpha = s/R = s/57.25$.
So $\theta = 0.265 - s/57.25$ … (2)
Hence from (1) and (2) we have
$h = 57.25[1 - \cos(0.265 - s/57.25)]$.
Check: When $s = 0$, $h = 2$ (rounded from 1.99)

To substitute this expression into the volume formula to obtain V as a function of ‘s’ is routine but challenging and becomes messy:
$V = \pi R^3(2 + \cos(\sin^{-1}(r/R) - s/R))(1 - \cos(\sin^{-1}(r/R) - s/R))^2$.

It is simpler to proceed in two stages: first calculate values of ‘h’ from chosen values of ‘s’.
Then obtain corresponding volumes using $V = \pi h^2 (3R - h)/3$.

s (m)	h (m)	V (m^3)	V/capacity (%)
0	2.00	710	100.00
1	1.74	542	76.36
2	1.51	406	57.14
3	1.29	297	41.77
4	1.09	211	29.72
5	0.90	145	20.46
6	0.73	96	13.55
8	0.45	36	5.08
10	0.23	9.8	1.38
15	0.00	0.00	0.00

Table 2 Dam volume and depth against distance dam water has receded from high water mark

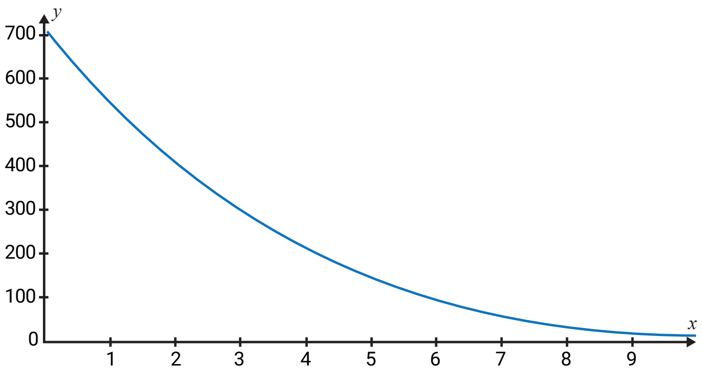

Figure 2 Graph of volume against 's' for circular dam

Solve the mathematics (a) for dam volume

(Note that 's' takes values from 0 to the length of arc (AM) = R(θ +α) ≈ 0.265 x 57.25 or about 15m.)

Example: s = 1 gives h = 1.745 and thence V ≈ 542 (m^3). This is approximately 76% of the dam capacity of 710 m^3. So when the distance to the water's edge has dropped by only 1 metre (approx 6.67%), the volume has been reduced by almost one-quarter.

Table 2 shows the results of calculating (using a spreadsheet) these quantities for values of 's' at (initially) one-metre intervals. Notice that when the waterline has receded only three metres from the top of the dam, the volume of water is much less than 50% of the capacity.

To interpret the results, Figure 2 contains the graph of $V = \pi R^3(2 + \cos(\sin^{-1}(r/R) - s/R))(1 - \cos(\sin^{-1}(r/R) - s/R))2$ generated by using graphing technology. This could be used, for example, as a chart pinned on the kitchen wall from which the volume of water remaining can be read, given only the distance (s) to the water line from the high water mark when the dam is full.

Formulate the mathematical model (b) for water loss over time

To estimate how long water in a dam will last, we need to make further assumptions – about evaporation.

Assumptions

Internet research will typically show that the evaporation rate varies with temperature, wind speed, sunshine, and relative humidity.

It also varies throughout the year, but a rough daily evaporation rate (average) can be found in (cm or mm) by dividing the average annual value by 365 days. This will be sufficient accuracy for estimation purposes, although in practice there will be seasonal variations.

So estimating the value for central Victoria, from data given in the problem description, we obtain the average amount of evaporation per day = 160/365 ≈ 0.44 (cm).

This is the 'depth' of water that is lost across any exposed surface area in a day. The volume lost will vary with the surface area.

Calculations will overestimate the number of days that suitable water is available to animals. Near the end the dam will resemble a bog and the water undrinkable.

Solve the mathematics (b) for water loss over time

The daily amount lost by evaporation will vary with the area of water surface, but by using the average value we can estimate how long the water should last without further rain.

From the data provided for this problem, for central Victoria we assume that the average amount of evaporation per day is approximately 160/365 ≈ 0.44(cm). This is the 'depth' of water that is lost across any exposed surface area in a day. The volume lost will vary with the surface area.

From Figure 1, the cross-sectional area when the depth of the dam is 'h' is a circle with radius 'x' and area πx^2 where $x^2 = 2Rh - h^2$.

Hence $A(h) = \pi(2Rh - h^2)$.

Note when h = 0, A = 0 and when h = 2,
A = 706.85 – the value of π $(15)^2$.

Mean value of the cross-sectional surface area averaged over the interval h = 0 to h = 2 is given by

$$\frac{1}{2-0}\int_0^2 \pi\,(2Rh - h^2)\,dh = \frac{2\pi}{3}\,(3R - 2) \approx 355.5, \text{ since } R = 57.5.$$

If the dam is full and no more rain falls, we can estimate how many days it would take for the dam to dry up:

710/1.56 ≈ 455 (approximately 65 weeks or 15 months worth of water supply).

If the dam must sustain a herd of 300 sheep on dry pasture, then daily consumption of water by 300 sheep (from the example data) is 2500 × 300/365 ≈ 2055 litres ≈ 2.055m^3.

Total average water loss per day from (consumption + evaporation) ≈ 3.62 m^3.

Number of days of water available ≈ 710/3.62 ≈ 196 days (about 28 weeks or just over 6 months).

Interpret the solution

Mathematical outcomes have been continually linked to the dam structure, its volume and dimensions, and practical implications – as for example the meaning of the graph in Figure 2. This is typical of problems involving a variety of mathematical calculations. Their meaning within the problem needs to be interpreted and assessed as they arise, for testing numerical outcomes against the real situation will often identify errors in calculation that need to be addressed.

The formula obtained translates the stepped out distance (s) into a corresponding value for volume that gives estimates of volume for any measured value of s. This would provide the basis for constructing a ready reckoner or wall chart if desired.

The data provided assumed a consistent evaporation rate based on an annual average. But the scenario posed — drought conditions — could be considered to be different from the average. How will the outcome vary, if different values are considered for the evaporation rate?

The data provided for livestock water requirements are annual averages. In a real-world scenario, are animals likely to need more water in drought conditions than in average weather? How will the outcome vary if different values are considered for the water consumption rate?

For water loss over time, noting the observation about boggy conditions when the dam is nearly empty, the figure obtained will overestimate the time drinkable water will be available. A safer estimate would be about one month less. Perhaps?

Students can contribute actively to these sorts of ideas, and resulting refinements, once they engage with properties of the real contextual setting.

Evaluate the model

Apart from the checking of working for possible errors in mathematical calculations and/or in the application of technology, evaluation involves continuous checking against the needs of the problem context.

Has the solution provided a sufficiently good answer to the problem posed, or do we need further work? Why?

Sometimes when the answer to the first question is 'yes' the first answer obtained suggests a deeper exploration that only becomes obvious from the initial modelling effort. This then stimulates another cycle of modelling with an amended purpose.

A different perspective on evaluation was reported by a teacher who used a version of this problem with her year 10 class. An appreciative parent who happened to be a farmer told her that stepping down the bank was the method he and others used in estimating the amount of water in a dam.

Report the solution

The modelling report could contain all the above components of the solution of the problem. It should summarise and illustrate how the mathematical insights obtained advanced an understanding of the problem — even if this sometimes means that the solution attempt in its present state is in need of further development. All assumptions and choices of data values should be explained and justified.

In this case the report should develop as a systematic and cohesive narrative: considering the implications of drought conditions for livestock farmers; providing tools for farmers to use to estimate water supply, such as the data shown in Table 2 and the graph in Figure 2; and recommending a timeframe within which provisions should be made for alternative supplies for stock.