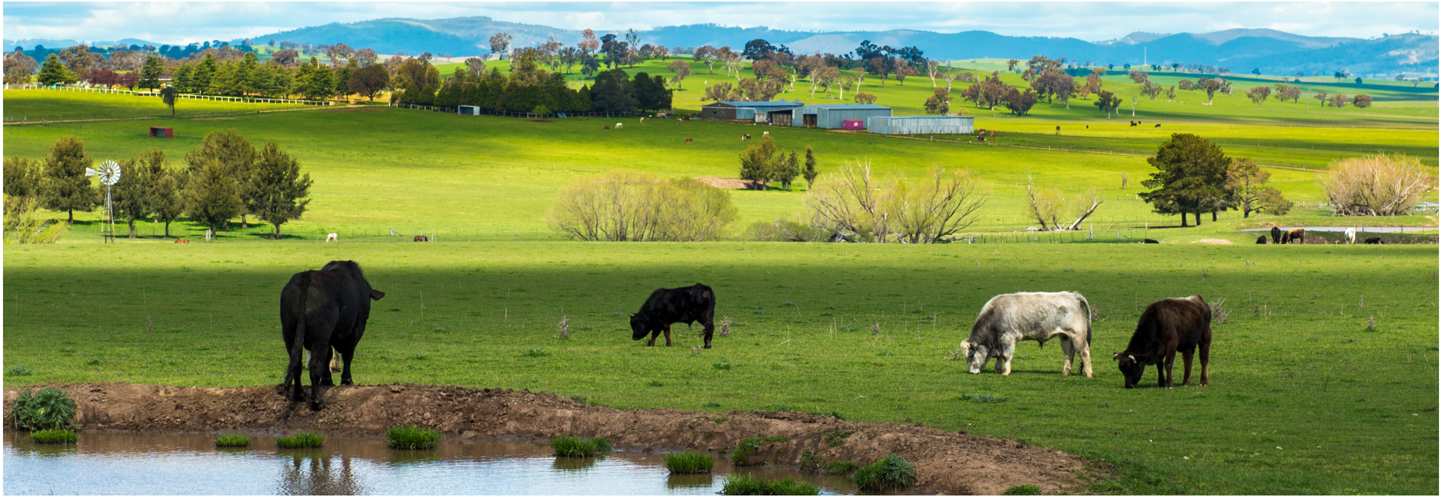
Example problem

Level: Middle secondary

Harder modelling

(IM²C international mathematical modeling challenge)

Farm dams



Describe the real-world problem

Farmers get their feet wet

In a dry country like Australia, farm dams are part of the lifeblood of rural life. Knowing the volume of water at any time is important for planning the numbers and distribution of livestock, and estimating when the supply is likely to run out under drought conditions.

Methods of estimating the volume of a partly empty dam depend on interpreting physically observable signs. If depth markers are embedded when the dam is excavated, the volume can be estimated from the water level measured on the marker. If dams do not have markers, some other method is needed.

Dams lose water by evaporation from the water surface to the atmosphere. Annual average evaporation rates are estimated using data collected from locations throughout the country. They vary from about 100 cm in western Tasmania up to 400 cm in the desert regions of northern Australia. Values in Victoria vary from about 140 cm in the south to 180 cm in the north, according to the Bureau of Meteorology (http://www.bom.gov.au/watl/evaporation). Loss of water by seepage is negligible.

Livestock water requirements

The table, containing data from a primary industries website, shows water requirements for a variety of farm animals. It contains information that farmers would know for their own stock, and is provided as a resource for this problem.

Table 1 Livestock water requirements

Stock	**Litres/animal/year**
Sheep	
nursing ewes on dry feed	3300
fat lambs on dry pasture	800
mature sheep — dry pasture	2500
fat lambs — irrigated pasture	400
mature sheep — irrigated pasture	1300
Cattle	
dairy cows, dry	16 000
dairy cows, milking	25 000
beef cattle	16 000
calves	8000

Specify the mathematical problems

Find a method of estimating the amount of water in a partly empty dam.

We have made an assumption that the dimensions of the dam when empty (as excavated) are known, and that they are as shown in Figures 1 and 2. The dam (Figure 1) has been constructed by excavating a horizontal square base ABTW. Sides ABCD and WTSR are vertical, while the ends ADRW and BCST have been sloped back to ground level so that the distances AD and BC are equal to half the length of AB (lengths are not to scale). Livestock gain access to the water in the dam via the sloping ends.

Figure 2 shows the vertical cross-section ABCD. The side length of the bottom square ABTW is '2a', which is also the dam width. The depth of the dam when full is 'd', and 'h' represents the depth when it is partly empty. Sloping lengths AD and BC are of length 'a'. For this dam, a = 10 and d = 2 where the distances are in metres.

Problem a: Model for dam volume

Find a way of estimating the volume of water in the dam when the (unknown) depth of water is 'h'.

Problem b: Model for water loss over time

Suppose the dam is in northern Victoria. If the dam is full and no more rain falls, how many days it would take for the dam to dry up? Now suppose that a herd of 100 beef cattle is sustained by the dam. If the dam is full and no more rain falls, how long will it be before other arrangements need to be made for the stock?

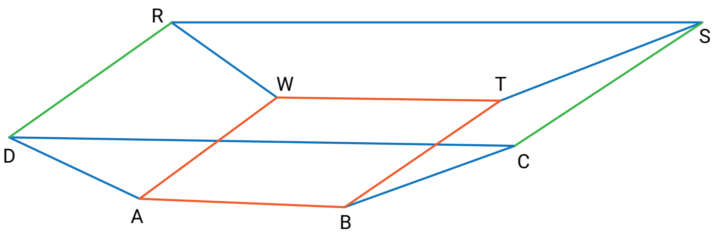


Figure 1 Diagram of empty rectangular dam

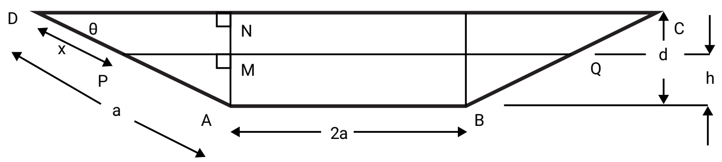


Figure 2 Vertical section through rectangular dam

Formulate the mathematical model (a)

Discussion with students can be used to identify variables and measurements that will be needed to address the problem. Only data that can be accessed are useful, and the depth of water is not one of these. One readily obtainable piece of data is the distance the water surface has receded from its position when the dam is full — a distance obtained simply by stepping out the distance taken to walk from the edge of the dam, directly to the current water level. A key insight is the recognition that this distance (x) is both measurable, and a key input to the model.

Assumptions

Dams with rectangular surface areas and with access for animals can be represented by trapezoidal volumes of various kinds. The dimensions of the dam when empty (as excavated) are known.

Model for dam volume

We need to create a formula that enables the volume to be estimated when the distance DP = x is known. (x is the distance found by stepping out or measuring the distance the water has receded down the slope from its highest point when the dam is full.)

Volume of dam when full = area (ABCD) × width (2a)

In trapezium ABCD: length CD = AB + 2x DN = 2a + 2acosθ where $\sin\theta = d/a$

Area of ABCD = xa (2 + cos θ) × d

Volume of dam (full) = 2 a^2d (2 + cos θ)

Given a = 10, d = 2, sin θ ≈ 0.2 (θ ≈ 11.5° and cos θ ≈ 0.98)

Volume of dam (full) = 1192m^3 (approx.) = 1.192 megalitres

To find volume of partly filled dam consider cross-section ABQP in Figure 2.

PM/MA = DN/NA (similar triangles) so PM/h = a cos θ/d and PM = ah cos θ/d.

Area (ABQP) = $\frac{1}{2}$ xh (4a + 2ah cos θ/d) so volume at height h is given by:

V(h) = area (ABQP) × 2a = 2a^2h[2 + (h/d) cos θ].

Hence V(h) = 200h(2 + 0.49h) – a quadratic in 'h'.

(Note as a check that when h = d = 2, V = 1192 as above.)

Now we need to express 'h' in terms of 'x'.

By similar triangles DNA and PMA:
(a – x)/h = a/d so h = (d/a)(a – x) = 0.2(10 – x).

Substitution in the formula for V(h) gives:
V(x) = 40(10 – x)(2.98 – 0.098x).

Check: When x = 0, V = 1192 as above.

This formula can now be written more tidily as:

V(x) = 3.92(10 – x) (30.4 – x) which rounds to a value of 1192 when x = 0.

Solve the mathematics (a)

Note that the constant checking of numerical values for consistency, during formulation, can also be considered part of the solution process. Volumes corresponding to different values of 'x' calculated from the formula above are shown in Table 2 below. Volume columns show the respective percentages of the total capacity. Also shown (for comparison) are the values of the inaccessible depth measures (h) that correspond to the different selected values of the accessible measure (x).

Figure 3 is a graph of $V = 3.92(10 - x)(30.4 - x)$ for values of x between 0 and 10.

It could be used, for example, as a chart pinned on the kitchen wall from which the volume of water remaining can be read, given only the distance (x) to the water line from the high water mark when the dam is full.

Table 2 Dam volume and depth against distance dam water has receded from high water mark

x (metres)	Volume (% of whole)	h (metres)
0	100	2.0
1	87.0	1.8
2	74.7	1.6
3	63.1	1.4
4	52.1	1.2
5	41.8	1.0
6	32.1	0.8
7	23.1	0.6
8	14.7	0.4
9	7.0	0.2
10	0	0

Formulate the mathematical model (b) for water loss over time

To estimate how long water in a dam will last, we need to make further assumptions – about evaporation.

Assumptions

Water in the dam is reduced by evaporation and in providing for livestock. Seepage loss is negligible and can be ignored.

Internet research will typically show that the evaporation rate varies with temperature, wind speed, sunshine, and relative humidity.

It also varies throughout the year, but a rough daily evaporation rate (average) can be found in (cm or mm) by dividing the average annual value by 365 days. This will be sufficient accuracy for estimation purposes, although in practice there will be seasonal variations.

Using the value for northern Victoria given in the problem description, we obtain the average amount of evaporation per day $= 180/365 \approx 0.5$(cm).

This is the 'depth' of water that is lost across any exposed surface area in a day. The volume lost will vary with the surface area.

Calculations will overestimate the number of days that suitable water is available to animals. Near the end the dam will resemble a bog and the water undrinkable.

Model for water loss over time

From Figure 2, the cross-sectional area when the depth of the dam is 'h' [A(h)] is a rectangle with length PQ and width 2a.

$A(h) = (2a + 2ah\cos\theta)/d \times 2a = 4a^2 (1 + h\cos\theta/d)$
$= 400(1 + 0.098h)$ when $a = 10$, $d = 2$.
When $a = 0$, $A = 400$ (area of the square base of the dam).

The area varies linearly with h, so the average cross-sectional area occurs when $h = 1$.
So average value of the exposed area of water surface $= 439.2\ m^2$, for which the daily evaporation loss would be
$439.2 \times (0.5)/100 \approx 2.2\ m^3$.

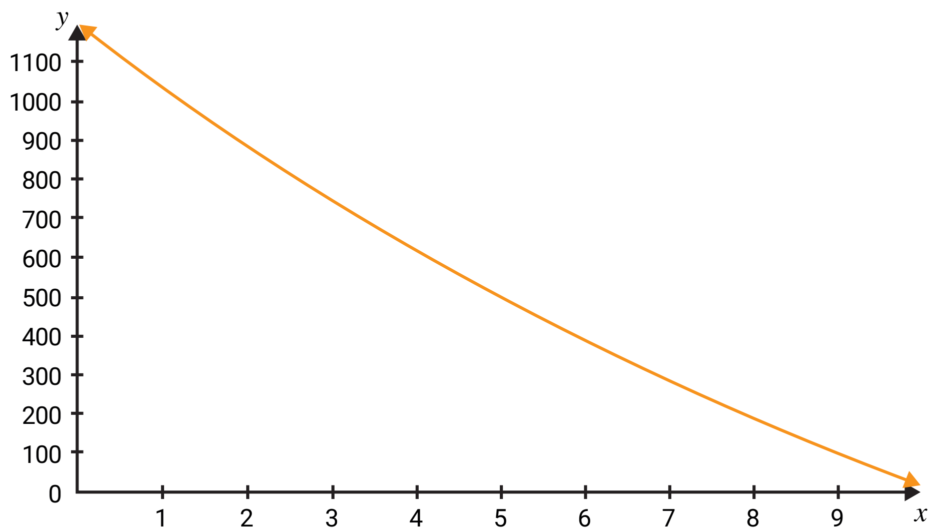


Figure 3 Graph of volume against x for rectangular dam

Solve the mathematics (b) for water loss over time

If the dam is full and no more rain falls, we can estimate how many days it would take for the dam to dry up:
1192/2.2 ≈ 541 days (approximately 18 months' supply).

If the dam must sustain a herd of 100 beef cattle, then daily consumption of water by 100 beef cattle (from the example data):
16000 × 100/365 ≈ 4384 litres ≈ 4.38m^3.

Total average water loss per day from (consumption + evaporation) ≈ 6.58 m^3.

Number of days of water available before other arrangements must be made to maintain livestock ≈ 1192/6.58 ≈ 181 days (about 26 weeks or 6 months).

Interpret the solution

Mathematical outcomes have been linked throughout to the dam structure, its volume and dimensions, and practical implications – as for example the meaning of the graph in Figure 3. This is typical of problems involving a variety of mathematical calculations. Their meaning within the problem needs to be interpreted and assessed as they arise, for testing numerical outcomes against the real situation will often identify errors in calculation that need to be addressed.

The formula obtained, translates the stepped out distance (x) into a corresponding value for volume that gives estimates of volume for any measured value of x. This would provide the basis for constructing a ready reckoner, or wall chart if desired.

The data provided, assumed a consistent evaporation rate based on an annual average. But the scenario posed — drought conditions — could be considered to be different from the average. How will the outcome vary, if different values are considered for the evaporation rate?

The data provided for livestock water requirements are annual averages. In a real-world scenario, are animals likely to need more water in drought conditions than in average weather? How will the outcome vary if different values are considered for the water consumption rate?

For water loss over time, noting the observation about boggy conditions when the dam is nearly empty, the figure obtained will overestimate the time drinkable water will be available. A safer estimate would be about 5 months.

Evaluate the model

Apart from the checking of working for possible errors in mathematical calculations and/or in the application of technology, evaluation involves continuous checking against the needs of the problem context.

Has the solution provided a sufficiently good answer to the problem posed, or do we need further work?

Sometimes when the answer to the first question is 'yes' the first answer obtained suggests a deeper exploration that only becomes obvious from the initial modelling effort. This then stimulates another cycle of modelling with an amended purpose.

A different perspective on evaluation was reported by a teacher who used a version of this problem with her Year 10 class. An appreciative parent who happened to be a farmer told her that stepping down the bank was the method he and others used in estimating the amount of water in a dam.

Report

The modelling report could contain all the above components of the solution of the problem. It should summarise and illustrate how the mathematical insights obtained advanced an understanding of the problem — even if this sometimes means that the solution attempt in its present state is in need of further development. All assumptions and choices of data values should be explained and justified.

In this case the report should develop as a systematic and cohesive narrative: considering the implications of drought conditions for livestock farmers; providing tools for farmers to use to estimate water supply, such as the data shown in Table 2 and the graph in Figure 3; and recommending a timeframe within which provisions should be made for alternative supplies for stock.